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1969 J. Phys. A: Gen. Phys. 2 262

(http://iopscience.iop.org/0022-3689/2/3/003)

J. PHYS. A (GEN. PHYS.), 1969, SER. 2, VOL. 2. PRINTED IN GREAT BRITAIN

General relativistic theory of Lagrangian functions of 'Bose' fields in scalar representation

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MS. received 30th December 1968

Abstract. A method is developed for the derivation of general relativistic Lagrangian funcrions of 'Bose' fields. The method consists of the solution of some functional equations which are based only on the requirements of general covariance. The most general exact solutions of the 'strong' equations are derived.

1. Introduction

In a recent paper (Knapecz 1969) it was proved that any geometric object $\Phi_{\mathrm{H}}(x)$ may be expressed as a concomitant of an adequate aggregate of scalar potentials $S^A(x)$, $A = 1, 2, \ldots, M$:

$$\Phi_{\mathrm{H}}(x) = F_{\mathrm{H}}\left[x^k, S^A(x), \frac{\partial S^A}{\partial x^l}, \frac{\partial^2 S^A}{\partial x^l\,\partial x^m}, \ldots\right] \tag{1}$$

where F_{H} obeys the transitivity and invertibility conditions (Nijenhuis 1952, Aczél and Golab 1960, Kucharzewski and Kuczma 1964). Since any field, which describes integer spin particles ('Bose' fields, $s = 0, \hbar, 2\hbar, \ldots$), are geometric objects (scalars, vectors, symmetric tensors, etc.) this theorem means that any 'Bose' field, as well as any system of them, may be described by some aggregate of scalar fields.

In the present paper we investigate what Lagrangian functions of 'Bose' fields are possible if the validity of the principle of general relativity is supposed.

In the next section we investigate the content of the principle of general relativity. In the following one we give the functional equations of Lagrangian functions. In the last two sections we deduce the most general Lagrangian densities of 'Bose' fields, if they are described by their scalar potentials.

Any theory of 'Bose' fields alone is not a complete one. It should be supplemented by the 'Fermi' fields of half-integer spin. Since very little is known about the geometric objects, which are suitable to describe the 'Fermi' fields, their theory will be developed in a series of publications, and thereafter the general relativistic Lagrangian theory of 'Fermi' fields will be treated.

2. The principle of general covariance

The content of a physical principle is to some extent a question of convention. If everybody applies the principle in the same sense, the convention exists. In the contrary case it should be described before its application.

Since different authors give different meaning to the principle of general relativity (Kretschmann 1917, Tolman 1934, Pauli 1958, Fock 1959, Epstein 1962, Kibble 1968) we describe what we understand by this principle.

By the principle of general covariance we understand two requirements:

(i) Nature and its phenomena are describable by geometric objects.

(ii) If the *complete* kinematical description of a system is realized by the *minimal aggregate* of geometric objects, then the dynamical (physical) laws of the system are expressible by this minimal aggregate, and the expressions of the laws are concomitants of the aggregate. The supplementation of the description by redundant variables (for example tetrads $V^A_k(x)$) or the introduction of the covariant derivatives (i.e. the introduction of the connections $\Gamma^i_{kl}(x)$), etc., is neither needed nor allowed. The concomitants of geometric objects are by themselves generally covariant.

The validity of these requirements is a question of experience. It is a fact that in the case of the theory of gravitation it was successfully applied. In this case the minimal aggregate consists of the metric tensor $g_{kl}(x)$, and the Lagrangian density $R(\det g_{ik})^{1/2}$ is one of the expressions which are constructed from the g_{ik} alone. No further variables are needed.

According to our opinion, the introduction of general covariant derivatives

$$\phi_{,k} \to \phi_{,k} - \Gamma_k \phi \tag{2}$$

into special relativistic expressions is a general relativistic *transcription* only, which has nothing to do with the principle of general covariance. That is, it is not certain that the transcription (2) in any case has a physical content. It cannot be a principle.

3. The fundamental equations

The Lagrangian functions are not observables, because they are given only up to a divergence: L and $L + A^s_{,s}$ are equivalent. Consequently L has no explicit transformation properties under a general coordinate transformation

$$\bar{x}^k = f^k(x^l), \qquad 0 \neq \det \frac{\partial \bar{x}^k}{\partial x^l} \neq \infty. \tag{3}$$

On the other hand, according to the principle of general relativity, the Eulerian equations of L should be geometric concomitants, since the field variables are geometric objects. Since the 'Bose' fields may be described by their scalar potentials $S^A(x^l)$, $A = 1, 2, \ldots, M$, in this representation the Eulerian equations should be either ordinary or Weyl densities of weight 1. Therefore the functional differential equations of Lagrangian functions of 'Bose' fields in scalar representation are either

$$\begin{aligned}
&\frac{\partial L[\bar{S}^B(\bar{x}), \bar{S}^B_{,\bar{k}}(\bar{x}), \ldots]}{\partial \bar{S}^A} - \frac{\partial}{\partial \bar{x}^s}\frac{\partial L[\bar{S}^B(\bar{x}), \ldots]}{\partial \bar{S}^A_{,\bar{s}}} + \ldots \\
&\quad = \left(\det \frac{\partial x^k}{\partial \bar{x}^l}\right)\left\{\frac{\partial L[S^B(x), S^B_{,k}(x), \ldots]}{\partial S^A} - \frac{\partial}{\partial x^s}\frac{\partial L[S^B(x), \ldots]}{\partial S^A_{,s}} + \ldots\right\}
\end{aligned} \tag{4}$$

or

$$\begin{aligned}
&\frac{\partial L[\bar{S}^B(\bar{x}), \bar{S}^B_{,\bar{k}}(\bar{x}), \ldots]}{\partial \bar{S}^A} - \frac{\partial}{\partial \bar{x}^s}\frac{\partial L[\bar{S}^B(\bar{x}), \ldots]}{\partial \bar{S}^A_{,\bar{s}}} + \ldots \\
&\quad = \left(\det \frac{\partial x^k}{\partial \bar{x}^l}\right)\left(\operatorname{sgn} \det \frac{\partial x^k}{\partial \bar{x}^l}\right)\left\{\frac{\partial L[S^B(x), S^B_{,k}(x), \ldots]}{\partial S^A} - \frac{\partial}{\partial x^s}\frac{\partial L[S^B(x), \ldots]}{\partial S^A_{,s}} + \ldots\right\}
\end{aligned} \tag{5}$$

which hold for arbitrary $S^A(x)$, x^k and $\bar{x}^k = f^k(x^l)$. The solutions of these equations contain the above-mentioned redundant divergence.

One may avoid the redundancy if one requires the Lagrangian to be an ordinary or a Weyl density of weight 1. In this way one obtains the 'strong' equations of Lagrangian functions. In the scalar representation the strong functional equations of the Lagrangian functions of 'Bose' fields are

$$\begin{aligned}
&L[\bar{S}^A(\bar{x}), \bar{S}^A_{,\bar{k}}(\bar{x}), \ldots, \bar{S}^A_{,\bar{k}^1\bar{k}^2\ldots\bar{k}^r}(\bar{x})] \\
&\quad = \left(\det \frac{\partial x^k}{\partial \bar{x}^l}\right) L[S^A(x), S^A_{,k}(x), \ldots, S^A_{,k^1k^2\ldots k^r}(x)]
\end{aligned} \tag{6}$$

or

$$\begin{aligned}
&L[\bar{S}^A(\bar{x}), \bar{S}^A_{,\bar{k}}(\bar{x}), \ldots, \bar{S}^A_{,\bar{k}^1\bar{k}^2\ldots\bar{k}^r}(\bar{x})] \\
&\quad = \left(\det \frac{\partial x^k}{\partial \bar{x}^l}\right)\left(\operatorname{sgn} \det \frac{\partial x^k}{\partial \bar{x}^l}\right) L[S^A(x), S^A_{,k}(x), \ldots, S^A_{,k^1k^2\ldots k^r}(x)].
\end{aligned} \tag{7}$$

4. Solutions of (6)

The 'Bose' field theories may be classified according to the multiplicity $M(S^A(x), A = 1, 2, \ldots, M)$ of their *minimal* scalar aggregate, according to the highest derivative r of $S^A(x)$, and according to the dimensionality n of the space–time. Therefore we denote by M_n^r the type of 'Bose' system.

Theorem 1. The most general Lagrangian of the type M^1 is

$$L = (\det S^k_{,l}(x)) I(S^A(x), I^\alpha_k) \tag{8}$$

where $k, l, \ldots, s, t, \ldots = 1, 2, \ldots, n$; $\alpha = n+1, n+2, \ldots, M$

$$I^\alpha_k = S^\alpha_{,t} T^t_k \tag{9}$$

$$T^k_s S^s_{,l} = \delta^k_l, \qquad T^s_l S^k_{,s} = \delta^k_l \tag{10}$$

and I is an invariant function of its M scalar and $n(M-n)$ invariant arguments I^α_k.

Proof. The fundamental equation (6) is in this case

$$L(\bar{S}^A, \bar{S}^A_{,k}) = \left(\det \frac{\partial x^i}{\partial \bar{x}^j}\right) L(S^A, S^A_{,k}). \tag{11}$$

Since S^A are scalars

$$\bar{S}^A = S^A \tag{12}$$

and

$$\bar{S}^A_{,\bar{l}} = S^A_{,t} \frac{\partial x^t}{\partial \bar{x}^l}. \tag{13}$$

At an arbitrary instant point P in some frame F the equation (13) for the fields $S^1, S^2, \ldots, S^n$ may take the form

$$S^k_{,t}(\mathrm{P}) \frac{\partial x^t(\mathrm{P})}{\partial \bar{x}^l(\mathrm{P})} = \delta^k_l, \qquad k, l = 1, 2, \ldots, n. \tag{14}$$

Since by supposition the scalars S^A are independent:

$$\Delta \equiv \det S^k_{,l} \neq 0 \tag{15}$$

we obtain from (14) that at the instant point P

$$\frac{\partial x^m(\mathrm{P})}{\partial \bar{x}^l(\mathrm{P})} = T^m_l(\mathrm{P}) \equiv \frac{\partial \ln \Delta(\mathrm{P})}{\partial S^l_{,m}(\mathrm{P})} \tag{16}$$

where T^m_k is the inverse of $S^k_{,m}$

$$T^k_t S^t_{,l} = \delta^k_l, \qquad S^k_{,t} T^t_l = \delta^k_l. \tag{17}$$

Inserting (16) into (11) we obtain (8), which was the result to be proved.

Theorem 2. The most general Lagrangian of the type M^2 is

$$L = (\det S^k_{,l}(x)) I(S^A(x), I^\alpha_k, I^\alpha_{kl}) \tag{18}$$

where the I_k are given by (9),

$$I^\alpha_{kl} = (S^\alpha_{,uv} - S^\alpha_{,t} T^t_\mu S^\mu_{,vw}) T^v_k T^u_l \tag{19}$$

and I is an invariant function of its S^A scalar, as well as its I_k and I_{kl} invariant arguments, whose numbers are respectively, M, $n(M-n)$ and $\binom{n+1}{2}(M-n)$.

Proof. The fundamental equation (6) in this case is

$$L(\bar{S}^A, \bar{S}^A_{,\bar{k}}, \bar{S}^A_{,\bar{k}\bar{l}}) = \left(\det \frac{\partial x^i}{\partial \bar{x}^j}\right) L(S^A, S^A_{,k}, S^A_{,kl}) \tag{20}$$

where the expressions of the second derivatives $\bar{S}^A_{,\bar{k}\bar{l}}$ are

$$\bar{S}^A_{,\bar{k}\bar{l}} = \left(S^A_{,t}\frac{\partial x^t}{\partial \bar{x}^k}\right)_{,\bar{l}} = S^A_{,st}\frac{\partial x^s}{\partial \bar{x}^k}\frac{\partial x^t}{\partial \bar{x}^l} + S^A_{,t}\frac{\partial^2 x^t}{\partial \bar{x}^k\,\partial \bar{x}^l}. \tag{21}$$

The number of the $\partial^2 x^m/\partial\bar{x}^k\partial\bar{x}^l$ and that of the $S^m_{,\bar{k}\bar{l}}$ $(m = 1, 2, ..., n)$ is equal. Since $S^m{}_{kl}$ has a non-homogeneous transformation rule, at an instant point P in some coordinate system F it may happen that

$$S^k_{,\bar{l}\bar{m}}(\mathrm{P}) = 0. \tag{22}$$

Then

$$S^k_{,t}(\mathrm{P})\frac{\partial^2 x^t(\mathrm{P})}{\partial \bar{x}^l\,\partial \bar{x}^k} = -S^k_{,tu}(\mathrm{P})\frac{\partial x^t(\mathrm{P})}{\partial \bar{x}^k}\frac{\partial x^u(\mathrm{P})}{\partial \bar{x}^l}. \tag{23}$$

Multiplying (23) on the left-hand side by T^u_k, and taking (16) into account, we obtain

$$\frac{\partial^2 x^u(\mathrm{P})}{\partial \bar{x}^k\,\partial \bar{x}^l} = -T^u_v(\mathrm{P})S^v_{,st}(\mathrm{P})T^s_k(\mathrm{P})T^t_l(\mathrm{P}). \tag{24}$$

Inserting (24) into (20), we obtain (18), by which the theorem is proved.

Theorem 3. The invariant I_{kl} of (19) may be represented by the form

$$I^\alpha_{kl} = (S^\alpha_{,u}T^u_k)_{,v}T^v_l = I^\alpha_{k,v}T^v_l. \tag{25}$$

Proof. Since

$$T^m_v S^v_{,tl}T^t_k = -T^m_{v,l}S^v_{,t}T^t_k = -T^m_{k,l} \tag{26}$$

we obtain

$$S^\alpha_{,mv}T^v_k - S^\alpha_{,t}T^t_u S^u_{,vm}T^v_k = S^\alpha_{,mv}T^v_k + S^\alpha_{,t}T^t_{k,m} = (S^\alpha_{,t}T^t_k)_{,m} \tag{27}$$

and the statement is proved.

We mention without proof that all invariants $I^\alpha_{k^1k^2\ldots k^r}$ of higher order may be represented by the form (25):

$$I^\alpha_{k^1k^2\ldots k^r} = (I^\alpha_{k^1k^2\ldots k^{r-1}})_{,t}T^t_{k^r}. \tag{28}$$

Therefore the Lagrangians of the type M^r_n are

$$L = (\det S^k_{,l}(x))I(S^A(x), I^\alpha_k, I^\alpha_{k^1k^2}, \ldots, I^\alpha_{k^1k^2\ldots k^r}). \tag{29}$$

5. Solutions of (7)

The Lagrangians of the Weyl kind are not applied in the physics. Without proof, we note that the most general Lagrangian density of the M^r_n type of the Weyl kind is

$$L = (\det S^k_{,l})(\mathrm{sgn}\det S^k_{,l})I(S^A, I^\alpha_k, \ldots, I^\alpha_{k^1\ldots k^r}). \tag{30}$$

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